Lecture 2 Intermediate Jacobian

§1 IJ

X smooth proj. $/\mathbb{C}$

$r \in \mathbb{N}$

- $H^{2r-1}(X,\mathbb{Z})$ is a weight $(2r-1)$ pure Hodge structure:
  - $H^{2r-1}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{\substack{p+q=2r-1 \\ p,q\geq 0}} H^{p,q}(X)$ where $H^{p,q}(X) \simeq H^q(X,\Omega_X^p)$
  - $H^{p,q}(X) = \overline{H^{q,p}(X)}$

- Equivalently, Hodge filtration

$$F^p H^{2r-1}(X,\mathbb{C}) := \bigoplus_{\substack{r,s \\ r\geq p}} H^{r,s}(X)$$

satisfies

$$\begin{cases} F^p \cap \overline{F^{2r-p}} = 0 \\ F^p \oplus \overline{F^{2r-p}} = H^{2r-1}(X,\mathbb{C}) \end{cases} \quad \forall\, 0 < p \leq 2r-1$$

r-th <u>intermediate jacobian</u> of X is

<u>Def</u> $$J^r(X) := \frac{H^{2r-1}(X,\mathbb{C})}{F^r H^{2r-1}(X) \oplus H^{2r-1}(X,\mathbb{Z})}$$

<u>Lemma 1</u> $J^r(X)$ is a complex torus of (complex) dimension $\frac{1}{2} b_{2r-1}(X)$.

Pf:

$$\overline{F^r H^{2r-1}(X)} \oplus F^r H^{2r-1}(X) = H^{2r-1}(X,\mathbb{C})$$

$$H^{2r-1}(X,\mathbb{C}) = \underbrace{H^{2r-1,0} \oplus H^{2r-2,1} \oplus \cdots \oplus H^{r,r-1}}_{F^r} \oplus \underbrace{H^{r-1,r} \oplus \cdots \oplus H^{1,2r-2} \oplus H^{0,2r-1}}_{\overline{F^r}}$$

Since $H^{2r-1}(X,\mathbb{Z})/_{tor} \subset H^{2r-1}(X,\mathbb{R})$

$\Rightarrow \quad H^{2r-1}(X,\mathbb{Z})/tor \cap F^r = 0$

$$\left(\forall \alpha \in H^{2r-1}(X,\mathbb{R}), \quad \alpha \in F^r \Rightarrow \alpha = \bar{\alpha} \in \overline{F^r} \Rightarrow \alpha \in F^r \cap \overline{F^r} = \{0\} \right).$$

Since $\dim_{\mathbb{R}} H^{2r-1}(X,\mathbb{R}) = \dim_{\mathbb{R}} F^r$

$\Rightarrow$ $H^{2r-1}(X,\mathbb{Z})/_{tor}$ is a lattice inside $\dfrac{H^{2r-1}(X,\mathbb{C})}{F^r H^{2r-1}(X)}$

□

Example 1 (Albanese)

X sm proj$/\mathbb{C}$ of dim n, $H^{2n-1}(X,\mathbb{C}) = H^{n,n-1}(X) \oplus H^{n-1,n}(X)$

$$J^n(X) = \frac{H^{2n-1}(X,\mathbb{C})}{H^{n,n-1}(X) \oplus H^{2n-1}(X,\mathbb{Z})} = \frac{H^{n-1,n}(X)}{H^{2n-1}(X,\mathbb{Z})}$$

Serre duality:

$$\begin{tikzcd} H^{n-1,n}(X) \arrow[r, phantom, "\simeq"] & H^{1,0}(X)^\vee \\ H^{2n-1}(X,\mathbb{Z}) \arrow[u, phantom, "\cup"] \arrow[r, leftrightarrow] & H_1(X,\mathbb{Z}) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$$\Rightarrow \quad J^n(X) = \frac{H^0(X,\Omega^1_X)^\vee}{H_1(X,\mathbb{Z})} =: \mathrm{Alb}(X)$$

Example 2 (Picard variety)

X sm proj$/\mathbb{C}$, $H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$

$$J^1(X) = \frac{H^1(X,\mathbb{C})}{H^{1,0}(X) \oplus H^1(X,\mathbb{Z})} = \frac{H^{0,1}(X)}{H^1(X,\mathbb{Z})} = \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})}$$

by $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1$

$$\begin{tikzcd} 0 \arrow[r] & H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathcal{O}_X) \arrow[r] \arrow[d, twoheadrightarrow] & H^1(X,\mathcal{O}_X^*) \arrow[r, "c_1"] \arrow[d, equal] & H^2(X,\mathbb{Z}) \\ & & \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})} \arrow[ur, hookrightarrow] & \mathrm{Pic}(X) \arrow[r, twoheadrightarrow] & NS(X) \arrow[u, hookrightarrow] \end{tikzcd}$$

$$\Rightarrow \quad J^1(X) = \mathrm{Pic}^0(X)$$

§2 Abel–Jacobi invariant

X sm proj $/\mathbb{C}$ of dim n.

$r \in \mathbb{N}_{>0}$

Soit $\gamma \in CH^r(X)$ tq $[\gamma] = 0 \in H^{2r}(X,\mathbb{Z})$

i.e. $\gamma \in CH^r(X)_{hom}$

<u>Déf</u> $\Phi : CH^r(X)_{hom} \longrightarrow J^r(X) = \dfrac{H^{2r-1}(X,\mathbb{C})}{F^r H^{2r-1}(X) \oplus H^{2r-1}(X,\mathbb{Z})/_{tors}}$

$$\gamma \longmapsto \int_\sigma$$

où $\gamma = \partial\sigma$, σ : top. chain of real dim $2(n-r)+1$

$H^{2r-1}(X,\mathbb{C}) \xleftrightarrow{\text{dual}} H^{2n-2r+1}(X,\mathbb{C})$

$$\Rightarrow J^r(X) \cong \left\{ x \in H^{2n-2r+1}(X,\mathbb{C}) \;\middle|\; \begin{array}{l} \langle x, F^r H^{2r-1}(X)\rangle = 0 \\ \langle x, H^{2r-1}(X,\mathbb{Z})\rangle = 0 \end{array} \right\}$$

$$= F^r H^{2r-1}(X)^{\perp} \cap H^{2r-1}(X,\mathbb{Z})^{\perp}$$

- Φ is well-defined
- $\int_\sigma$ viewed as in $H^{2n-2r+1}(X)^*$, is in $F^r H^{2r-1}(X)^{\perp}$

$$0 \to F^r H^{2r-1}(X) \to H^{2r-1}(X) \to \frac{H^{2r-1}(X)}{F^r H^{2r-1}(X)} \to 0$$

dual:

$$\begin{tikzcd}
0 \arrow[r] & \left(\frac{H^{2r-1}(X)}{F^r H^{2r-1}(X)}\right)^* \arrow[r] \arrow[d, equal] & H^{2r-1}(X)^* \arrow[r] \arrow[d, equal, "\text{Poincaré}"] & F^r H^{2r-1}(X)^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F^r H^{2r-1}(X)^{\perp} \arrow[r] \arrow[d, equal] & H^{2n-2r+1}(X) \arrow[r] & F^r H^{2r-1}(X)^* \arrow[r] & 0 \\
& F^{n-r+1} H^{2n-2r+1}(X) . & & &
\end{tikzcd}$$

$$\Rightarrow \quad J^r(X) \cong \frac{F^{n-r+1} H^{2n-2r+1}(X)^*}{H_{2n-2r+1}(X, \mathbb{Z})} .$$

and $\Phi : CH^r(X)_{hom} \longrightarrow J^r(X) = F^{n-r+1} H^{2n-2r+1}(X)^* / H_{2n-2r+1}(X, \mathbb{Z})$

$\gamma \longmapsto \left([\eta] \mapsto \int_\sigma \eta \right)$ où $\partial\sigma = \gamma$.

<u>Lemma</u> Φ is well-defined:

Pf: • If take σ' another chain s.t. $\partial\sigma' = \gamma$

then $\partial(\sigma - \sigma') = 0 \Rightarrow \sigma - \sigma' \in H_{2n-2r+1}(X, \mathbb{Z})$

• If $[\eta] = [\eta']$ another representative of the cohom. class

then $\eta - \eta' = d\psi$ with ψ of type $(n-r+1, n-r-1) + \cdots \in F^{n-r+1} A^{2n-2r}(X)$.

$$\Rightarrow \int_\sigma d\psi = \int_{\partial\sigma} \psi = \int_\gamma \psi = 0$$

□

Example 1 (Abel)

$L \in \mathrm{Pic}(X)$ is trivial

$\Leftrightarrow$ it is topologically trivial and has A-J inv 0.

Example 2 $A_0(X) \xrightarrow{\text{alb}} \mathrm{Alb}(X)$ is the AJ map for $r=n$.

$$CH_0(X)_{\text{hom}} \xrightarrow{\Phi} J^n(X).$$